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Dynamics of Solitons in Coupled System of Scalar Fields

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The dynamics of interacting solitons of a system of two coupled nonlinear partial differential equations is studied numerically using a finite difference method in bi-dimensional spacetime. Stable, static topological solitons are obtained from an iterative-variational method, and used as initial solutions for the dynamical calculations. Some of the static solutions decay to stable solitons. Some subtle aspects of topological charges for the system under consideration are also discussed.

KEY WORDS: solitons; topological charges; coupled nonlinear PDEs.

1. INTRODUCTION

Coupled system of scalar fields have been studied by many authors (see Bazeia *et al.*, 1999 and references therein). One motivation for investigating these systems is the appearance of topological solitons similar to defects studied in cosmology and condensed matter physics (Vilenkin and Shellard, 1994). By solitons, we mean solutions that (i) have a finite and nonzero rest energy and (ii) are confined in a finite region in space at all times (i.e., non-dispersive) (Friedberg *et al.*, 1976).

Relativistic systems containing two scalar fields interacting with each other via a potential are particularly interesting, since there exists a consistent and covariant formulation for various physical quantities like energy, momentum and topological charges. Depending on the form of the potential V, the spectrum of solitons and their interactions can become quite rich. The resulting Euler-Lagrange equations are nonlinear coupled second-order partial differential equations that cannot be generally integrated analytically. Riazi *et al.* (2003) studied such a system in analogy with stable and unstable elementary particles. In this paper, we will present further results for the system introduced in (Riazi *et al.*, 2001).

Since the potential of this system can be written in a quadratic form, following the method of Bazeia *et al.* (1996, 1997), we obtained exact static solutions, and the

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stability and quantization of static solutions were investigated there. In this paper, using a variational method, we first obtain static solutions, and then consider these solutions as initial conditions for the dynamical equations. A suitable definition for the topological charges of the system is introduced, which exploits the topological structure of the vacuum space. The stability and interaction of soliton solutions is investigated numerically.

2. COUPLED SYSTEM OF SCALAR FIELDS

The Lagrangian density describing the system of two coupled real scalar fields is

$$\mathcal{L} = \frac{1}{2}\partial^{\mu}\phi\partial_{\mu}\phi + \frac{1}{2}\partial^{\mu}\chi\partial_{\mu}\chi - V(\phi,\ \chi), \tag{1}$$

where the potential $V(\phi,\ \chi)$ is a scalar function of the two fields ϕ and χ, bounded from below. It is well known that a potential having disconnected vacua leads to the appearance of so-called topological solitons in $1+1$ spacetime dimensions. We assume the potential to be in the form (Riazi *et al.*, 2001),

$$V(\phi,\ \chi) = \frac{1}{2}b^2\sin^2(b\phi)(1-\cos(c\chi))^2 + \frac{1}{2}c^2\sin^2(c\chi)(1-\cos(b\phi))^2, \tag{2}$$

in which b and c are real and positive constants. The motivation for considering this particular potential was discussed in (Riazi *et al.*, 2001). In particular, as shown in (Bazeia *et al.*, 1997), potentials having the form

$$V(\phi,\ \chi) = \frac{1}{2}H_{\phi}^2 + \frac{1}{2}H_{\chi}^2 \tag{3}$$

lead to the following first order differential equations for static configurations:

$$\frac{d\phi}{dx} = H_{\phi}, \tag{4}$$

and

$$\frac{d\chi}{dx} = H_{\chi}, \tag{5}$$

where

$$H(\phi,\ \chi) = (1-\cos(b\phi))\,(1-\cos(c\chi))\,. \tag{6}$$

The equations of motion obtained from the Lagrangian density (1) in bi-dimensional spacetime are

$$\Box\phi = f(\phi,\ \chi), \tag{7}$$

and

$$\Box\chi = g(\phi,\ \chi), \tag{8}$$

in which $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$, $\eta^{\mu\nu} = \mathrm{diag}(-1,\ 1)$, and

$$f(\phi,\ \chi) = \sin(b\phi)\cos(b\phi)\,(1-\cos(c\chi))^2 + b\sin^2(c\chi)\sin(b\phi)\,(1-\cos(b\phi))\,, \tag{9}$$

and

$$g(\phi,\ \chi) = \sin(c\chi)\cos(c\chi)\,(1-\cos(b\phi))^2 + c\sin^2(b\phi)\sin(c\chi)\,(1-\cos(c\chi))\,. \tag{10}$$

The classical vacuum of this system is the space of all points on the $(\phi,\ \chi)$ plane which satisfy $V(\phi,\ \chi) = 0$. This space consists of horizontal and vertical lines

$$b\phi = 2n\pi, \quad \text{for all } \chi, \tag{11}$$

and

$$c\chi = 2n\pi, \quad \text{for all } \phi \tag{12}$$

and discrete points in the $(\phi,\ \chi)$ plane

$$b\phi = (2n+1)\pi, \quad \text{and} \quad c\chi = (2m+1)\pi, \tag{13}$$

and

$$c\chi = (2n+1)\pi, \quad \text{and} \quad b\phi = (2m+1)\pi, \tag{14}$$

in which m and n are integers (see Fig. 1). In order to have localized, finite energy, solitons, the solutions should start and end at one of the vacuum points (or lines) mentioned above (Rajaraman, 1979, 1982). We consider the above restriction on

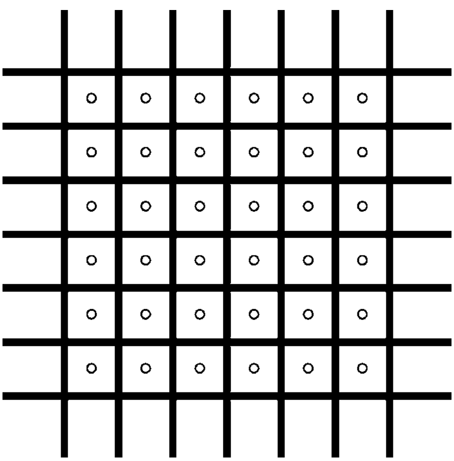

Fig. 1. The vacua on the $(\phi,\ \chi)$ plane consists of horizontal and vertical lines 2π apart, and isolated points at $((2n+1)\pi,\ (2m+1)\pi)$ (small circles).

solutions as boundary conditions for static and dynamic solutions. We have already obtained a few analytical solutions in (Riazi *et al.*, 2001). In the following section, we will employ a numerical scheme to calculate static solutions which cannot be obtained analytically.

3. STATIC SOLUTIONS

We find the static solutions using an iterative-variational method. Iterative methods consist of repeated application of an often simple algorithm. They yield a solution arbitrarily close to the exact, minimum energy solution, only as a sequence, even without consideration of round-off errors. In any iteration, one begins with an initial approximation and then successively modifies the approximation according to some rule (Ames, 1992). Here, we use the energy functional

$$E = \int_{-\infty}^{+\infty} T_o^o \, dx \tag{15}$$

as the quantity to be minimized. The energy-momentum tensor is defined in the usual way:

$$T^{\mu\nu} = \frac{1}{2}\partial^\mu\phi\partial^\nu\phi + \frac{1}{2}\partial^\mu\chi\partial^\nu\chi - \eta^{\mu\nu}\mathcal{L}. \tag{16}$$

We start with simple functions for $\phi(x)$ and $\chi(x)$ which begin and end at vacuum positions in the $(\phi, \ \chi)$ plane. For example, an initial approximation which satisfies these boundary conditions is

$$\phi(x) = \chi(x) = 2\arctan(e^{2x}), \tag{17}$$

corresponding to a transition from $(0, 0)$ to $(\pi/2, \ \pi/2)$ in the vacuum space of $(\phi, \ \chi)$. A program written in the MATLAB environment, successively modifies these initial functions, until the energy functional (15) reaches its minimum. The final solution obtained in this way is the same as our exact solution we obtained in (Riazi *et al.*, 2001) analytically, and this ensures the reliability of the numerical method. Accordingly, we seek for other static solutions which cannot be obtained analytically. These correspond to various transitions between different vacua

$$(n\pi, \ m\pi) \rightarrow (n'\pi, \ m'\pi), \tag{18}$$

where $n, \ m, \ n', \ m'$ are integers. Sample results are shown in Figs. 2 and 4. Note that the single soliton solution is similar to the kinks of the conventional sine-Gordon equation. However, the solution is not symmetric w.r.t. the center of the soliton (see Fig. 2 and 3).

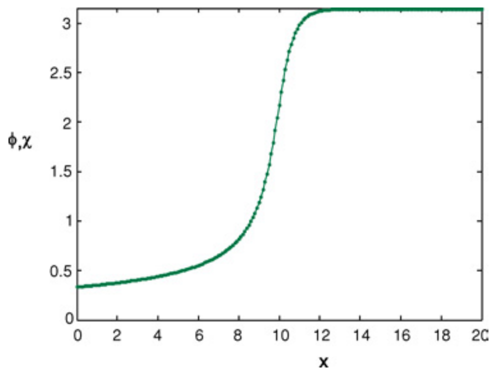

Fig. 2. The static solution corresponding to the transition $(0,\ 0) \to (\pi,\ \pi)$, obtained using the iterative method described in the text. $a = b = 1$ has been assumed. The solutions for ϕ and χ coincide. Note that contrary to the SG solitons, this soliton is not symmetric with respect to the center of the soliton.

4. TOPOLOGICAL CHARGES

Since the vacuum contains disconnected parts, topological charges and currents can be defined in analogy with the SG system. At first sight, two independent charges can be defined here, since there are two scalar fields:

$$J^{\mu}_{\phi} = \frac{b}{\pi}\epsilon^{\mu\nu}\partial_{\nu}\phi, \tag{19}$$

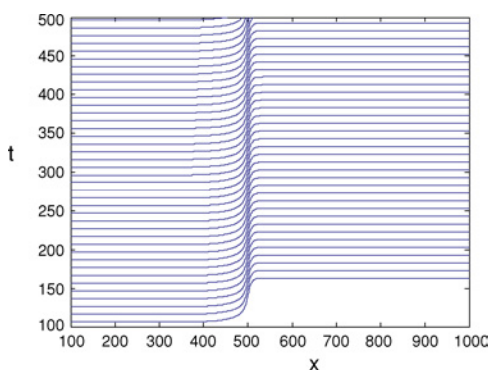

Fig. 3. The time evolution of the static solution shown in Fig. 2. No appreciable change is observed, indicating the dynamical stability of the soliton.

and

$$J^{\mu}_{\chi} = \frac{c}{\pi}\epsilon^{\mu\nu}\partial_{\nu}\chi, \tag{20}$$

with $\partial_{\mu}J^{\mu}_{\phi,\,\chi} = 0$. The corresponding conserved charges are

$$Q_{\phi} = \frac{b}{\pi}\int_{-\infty}^{+\infty}\frac{\partial\phi}{\partial x}dx, \tag{21}$$

and

$$Q_{\chi} = \frac{c}{\pi}\int_{-\infty}^{+\infty}\frac{\partial\chi}{\partial x}dx. \tag{22}$$

The soliton corresponding to Fig. (2), for example, has $Q_{\phi} = 1$ and $Q_{\chi} = 1$. One subtle point about the topological charges in the present and similar systems is the following: As can be seen from Fig. 1, the vacua corresponding to the horizontal and vertical lines are all connected to each other. Let us call the space of these connected points of vacuum Σ. Therefore, solutions beginning and ending at these vacua are trivial from a homotopical point of view (Williams, 1970; Patani *et al.*, 1976). The only non-trivial charges correspond to solutions which begin or end at isolated points indicated in Figure 1. From a topological point of view, all points on the connected part of the vacuum space (Σ) can be mapped into a single point, which we take it to be the $(0,\ 0)$ point on the $(\phi,\ \chi)$ plane. Moreover, by a continuous, finite-energy change in the fields, $Q_{\phi} = \pm 1$ and $Q_{\chi} = \pm 1$ solutions can be transformed into each other. We conclude that the common definitions (22) and (20) should be modified significantly for the system under consideration. Similar to the sine-Gordon equation, the potential is periodic in the ϕ and χ fields:

$$V(\phi + 2n\pi/b,\ \chi + 2m\pi/c) = V(\phi,\ \chi), \tag{23}$$

where m and n are arbitrary integers. Therefore, all points of the vacuum space can be transformed into each other, using elements of a discrete group of translations. We can enumerate these disconnected points by a pair of indices $(i,\ j)$ where

$$\phi_i = (2i + 1)\pi \tag{24}$$

and

$$\chi_j = (2j + 1)\pi. \tag{25}$$

An infinite number of conserved topological charges can be defined, which correspond to different values of i and j. The proper definition for the topological charges is therefore $Q_{ij} = 1$ if there is a transition from Σ to the disconnected vacuum point $(\phi_i,\ \chi_j)$. For the reverse transition we have $Q_{ij} = -1$. The solution depicted in Fig. 2, therefore, has $Q_{00} = 1$ and all other Q_{ij}'s equal to zero. For Fig. 4 we have $Q_{-1,\,-1} = 1$ and $Q_{00} = 1$ and all other Q_{ij}'s equal to zero.

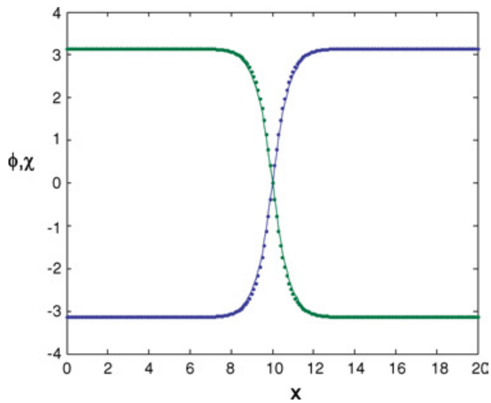

Fig. 4. The static solution corresponding to the transition $(\pi, -\pi) \to (-\pi, \pi)$, obtained using the iterative method described in the text. $a = b = 1$ has been assumed. The solutions for ϕ and χ are shown as solid curves.

5. DYNAMICAL SOLUTIONS

The coupled equations of motion (7) and (8) are hyperbolic second order PDEs. In order to solve them, we use a finite difference method described in (Riazi and Gharaati, 1998). In this method we take a rectangular box in the bi-dimensional spacetime and divide it into a net with constant intervals $x_i = i\epsilon$ and $t_j = j\delta$, in which i and j are integers and ϵ and δ are very small space and time intervals. Starting with the initial values of the scalar fields at $j = 1$ and $j = 2$, the following expressions are used successively to calculate the fields at later times:

$$\phi_{i,\,j+1} = \frac{\delta^2}{\epsilon^2}(\phi_{i+1,\,j} - 2\phi_{i,\,j} + \phi_{i-1,\,j}) + 2\phi_{i,\,j} - \phi_{i,\,j-1} - \delta^2 f_{i,\,j}, \tag{26}$$

and

$$\chi_{i,\,j+1} = \frac{\delta^2}{\epsilon^2}(\chi_{i+1,\,j} - 2\chi_{i,\,j} + \chi_{i-1,\,j}) + 2\chi_{i,\,j} - \chi_{i,\,j-1} - \delta^2 g_{i,\,j}, \tag{27}$$

where

$$\begin{aligned} f_{i,\,j} &= \sin(b\phi_{i,\,j})\cos(b\phi_{i,\,j})(1 - \cos(c\chi_{i,\,j}))^2 \\ &\quad + b\sin^2(c\chi_{i,\,j})\sin(b\phi_{i,\,j})(1 - \cos(b\phi_{i,\,j})), \end{aligned} \tag{28}$$

and

$$\begin{aligned} g_{i,\,j} &= \sin(c\chi_{i,\,j})\cos(c\chi_{i,\,j})(1 - \cos(b\phi_{i,\,j}))^2 \\ &\quad + c\sin^2(b\phi_{i,\,j})\sin(c\chi_{i,\,j})(1 - \cos(c\chi_{i,\,j})). \end{aligned} \tag{29}$$

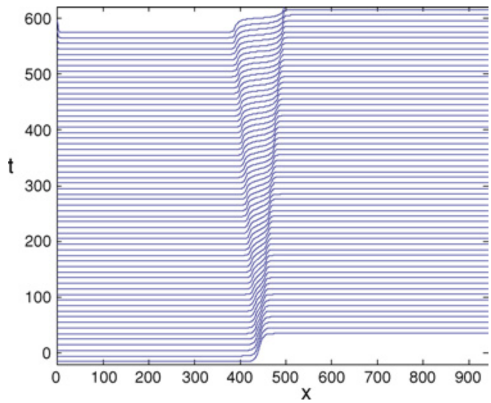

Fig. 5. The time evolution of the static solution shown in Fig. 4. The initially static solution spontaneously decay into two stable solitons.

In each calculation, we first obtain the static solution using the iterative method described in the previous section and then use these solutions as initial conditions in the above algorithm to obtain the dynamical solutions using finite difference method. What we have observed is the following: Static solutions fall into two categories: Stable and unstable. Stable solutions do not evolve with time and the dynamical results show very slight variations with time. Unstable solutions, on the other hand, evolve rapidly with time and split into stable, single soliton solutions. Figs. 4 and 5 show examples of both types. The energy density diagram corresponding to Fig. 5 is shown in Fig. 6.

6. SUMMARY

We investigated a system of coupled real scalar fields in bi-dimensional space-time. We obtained the static and dynamic solutions numerically, using iterative-variational and finite difference methods, respectively. From the time evolution of dynamical solutions we verified the stability of some of the static solutions. We also found the "decay products" of unstable solutions and we saw that after leaving the interaction region, they resume their shapes, with the extra energy converting into the kinetic energy of the daughter solitons. Topological charges corresponding to various solutions were discussed according to the structure of the vacuum space. One subtle point about the vacuum was the existence of a connected part (horizontal and vertical lines in the $(\phi,\ \chi)$ space or Σ as we called it), and isolated points in between. The result is that only solutions beginning or ending at the isolated points are stable. The numerical method we used is quite

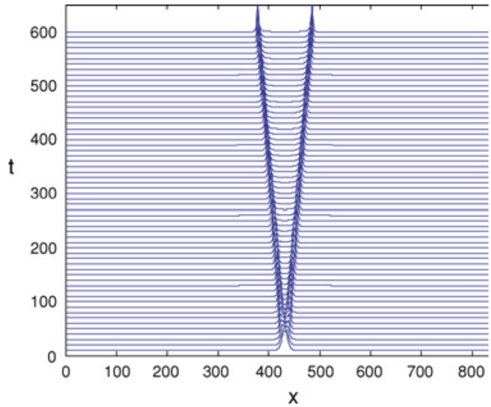

Fig. 6. The time evolution of the energy density corresponding to the decay of an initially static soliton $(-\pi, \pi) \rightarrow (\pi, -\pi)$. The daughter solitons are $(-\pi, \pi) \rightarrow (0, 0)$ and $(0, 0) \rightarrow (\pi, -\pi)$.

general and seems to be capable of calculating the interaction of three or more soliton systems.

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